Boundary-Layer Flow Over a Flat Plate

The Blasius Boundary-Layer Equations

f'''+0.5ff''

This can be re-writen as:

$f_3 = 0.5 \cdot f_0 \cdot f_2$

Since
f''' = f3
f'' = f2
f' = f1
f = f0

Where

$f_1 = \frac{u}{U} = \frac{d}{d\eta} f(\eta) = \frac{d}{d\eta} f_0$

and

$\frac{v}{U} = 0.5 \cdot \sqrt{\frac{\upsilon}{x \cdot U}} \cdot (n \cdot f_1 - f_0)$

and

$\eta = y \cdot \sqrt{\frac{U}{\upsilon \cdot x}}$

Here
U - Free Stream Velocity (m/s)
u - Component of velocity parallel with stream (m/s)
v - Component of velocity normal with stream (m/s)
υ - Kinematic Viscosity (kg/ms)
x - Distance along plate (m)
y - Distance normal to plate (m)
n - Non-dimensional coordinate

Boundary Conditions

Initial	Final
v = 0 ---> f0 = 0	u = 0 ---> f1=1
u = 0 ---> f1=0	
Guess, f2 = 0.5	

ORIGIN := 1

f2 := 0.5   This is a guess

f1 := 0   f0 := 0

$f(f2guess) := \begin{pmatrix} f2guess \\ f1 \\ f0 \end{pmatrix}$

Vector Boundary Conditions

$$f(f2) = \begin{pmatrix} 0.5 \\ 0 \\ 0 \end{pmatrix}$$

Derivative Vector

$$D(\eta, f) := \begin{pmatrix} -0.5 \cdot f_3 \cdot f_1 \\ f_1 \\ f_2 \end{pmatrix}$$

Now using, Mathcad's built-in Runge Kutta solver, the Blasius equations can be solved by iterating for accross a specfied range of the non-dimensionless paramter, η. The rkadapt function is used which is similar to rkfixed except internally uses adaptable spacing instead of fixed spacing.

$\eta start := 0$

$\eta end := 10$

$num\_steps := 2000$

$Z(f2guess) := Rkadapt(f(f2guess), \eta start, \eta end, num\_steps, D)$

$best := root\left(Z(f2)_{num\_steps+1, 3} - 1, f2\right)$

$Zfinal := Rkadapt(f(best), \eta start, \eta end, num\_steps, D)$

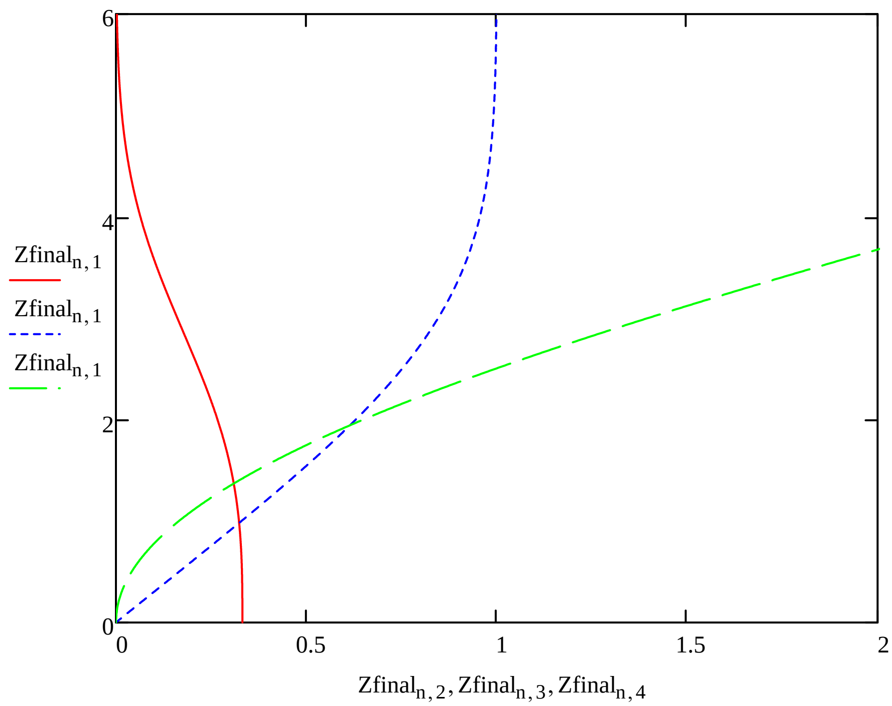
Zfinal =

	η	f2	f1	f0
	1	2	3	4
1	0	0.332	0	0
2	$5 \cdot 10^{-3}$	0.332	$1.66 \cdot 10^{-3}$	$4.151 \cdot 10^{-6}$
3	0.01	0.332	$3.321 \cdot 10^{-3}$	$1.66 \cdot 10^{-5}$
4	0.015	0.332	$4.981 \cdot 10^{-3}$	$3.736 \cdot 10^{-5}$
5	0.02	0.332	$6.641 \cdot 10^{-3}$	$6.641 \cdot 10^{-5}$
6	0.025	0.332	$8.301 \cdot 10^{-3}$	$1.038 \cdot 10^{-4}$
7	0.03	0.332	$9.962 \cdot 10^{-3}$	$1.494 \cdot 10^{-4}$
8	0.035	0.332	0.012	$2.034 \cdot 10^{-4}$
9	0.04	0.332	0.013	$2.656 \cdot 10^{-4}$
10	0.045	0.332	0.015	$3.362 \cdot 10^{-4}$
11	0.05	0.332	0.017	$4.151 \cdot 10^{-4}$
12	0.055	0.332	0.018	$5.022 \cdot 10^{-4}$
13	0.06	0.332	0.02	$5.977 \cdot 10^{-4}$
14	0.065	0.332	0.022	$7.015 \cdot 10^{-4}$
15	0.07	0.332	0.023	$8.135 \cdot 10^{-4}$
16	0.075	0.332	0.025	$9.339 \cdot 10^{-4}$

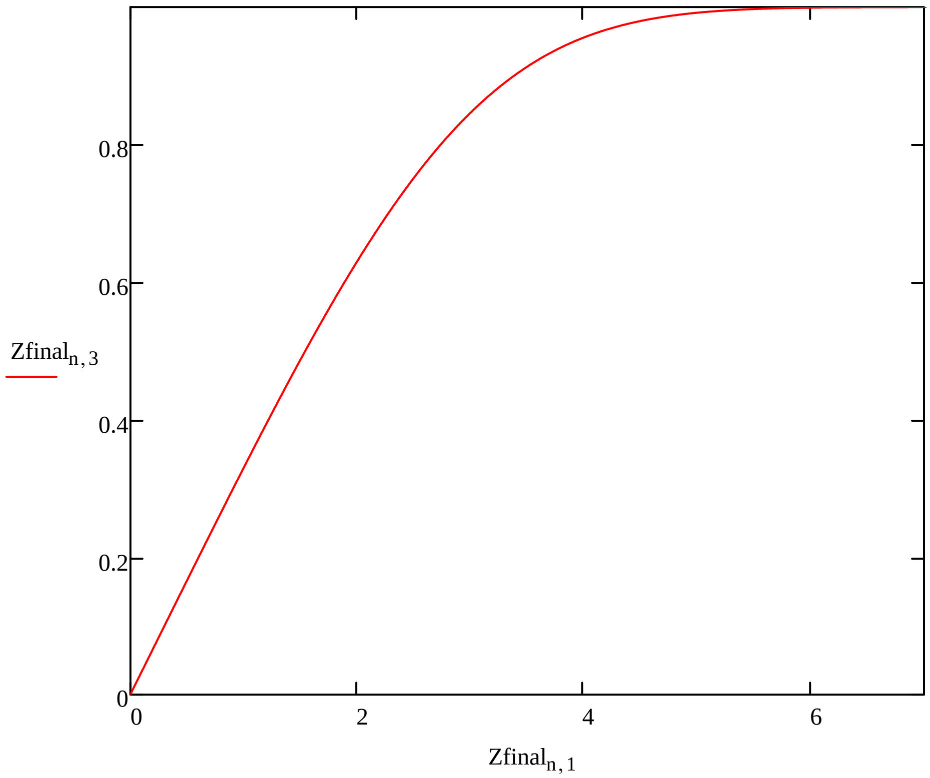
17	0.08	0.332	0.027	...

n := 1 .. num_steps

The Blasius solutions plotted against varying dimensionless parameter, η



Velocity distribution in the boundary layer along a flat plate



INPUT PARAMETERS

Free Stream Velocity	U (m/s)	$U := 260$
Density	ρ (kg/m3)	$\rho := 0.364$
Dynamic Viscosity	μ (kg/ms)	$\mu := 1.432 \times 10^{-5}$
Length of plate	L (m)	$L := 6.6$
Height of flow field	H (m)	$H := 0.2$
Wingspan	W (m)	$W := 79.8$
Wing wetted area	S (m)	$S := 2 \cdot L \cdot W$
		$S = 1.053 \times 10^{3}$
Kinematic Viscosity	υ (m2/s)	$\upsilon := \frac{\mu}{\rho}$
		$\upsilon = 3.934 \times 10^{-5}$

Assume transition occurs a Re=2x10^5

$Re_{crit} := 5 \times 10^{5}$

Transition point is

$$xcr := \frac{(Re_{crit} \cdot \upsilon)}{U}$$

$xcr = 0.076$ $\quad$ $\% := \left(\frac{xcr}{L}\right) \cdot 100$ $\quad$ $\% = 1.146$

The **Reynolds number** for a fully turbulent flow

$$Re := \frac{(U \cdot L)}{\upsilon}$$

$Re = 4.362 \times 10^{7}$ $\quad$ **Note:For Blasius equation to hold true, Re < 5 x 10^5**

The corresponding skin friction coefficient & drag for fully turbulent wing

$$Cf := \frac{0.455}{(\log(Re))^{2.58}}$$

$$Cf = 2.397 \times 10^{-3}$$

$$Dr := 0.5 \cdot \rho \cdot U^{2} \cdot S \cdot Cf$$

$$Dr = 3.107 \times 10^{4}$$

Over the leading edge, there is a region of laminar flow where the skin friction coefficient is

$$Cf_{l} := \frac{1.328}{\sqrt{Re_{crit}}}$$

$$Cf_{l} = 1.878 \times 10^{-3}$$

The wetted area therfore becomes

$$S_{l} := 2W \cdot xcr$$

$$S_{l} = 12.075$$

$$Dr_{l} := 0.5 \cdot \rho \cdot U^{2} \cdot S_{l} \cdot Cf_{l}$$

$$Dr_{l} = 278.999$$

If the flow was turbulent in this region, the skin friction coefficient would be

$$Cf_{t} := \frac{0.455}{\left(\log(Re_{crit})^{2.58}\right)}$$

$Cf_t = 5.106 \times 10^{-3}$

$Dr_t := 0.5 \cdot \rho \cdot U^2 \cdot S_l \cdot Cf_t$

$Dr_t = 758.483$

Overall Skin Fricion Drag and Coefficient

$Drag := Dr - \left(Dr_t - Dr_l\right)$

$Drag = 3.059 \times 10^4$

$$Cd := \frac{Drag}{0.25 \cdot \rho \cdot U^2 \cdot S}$$

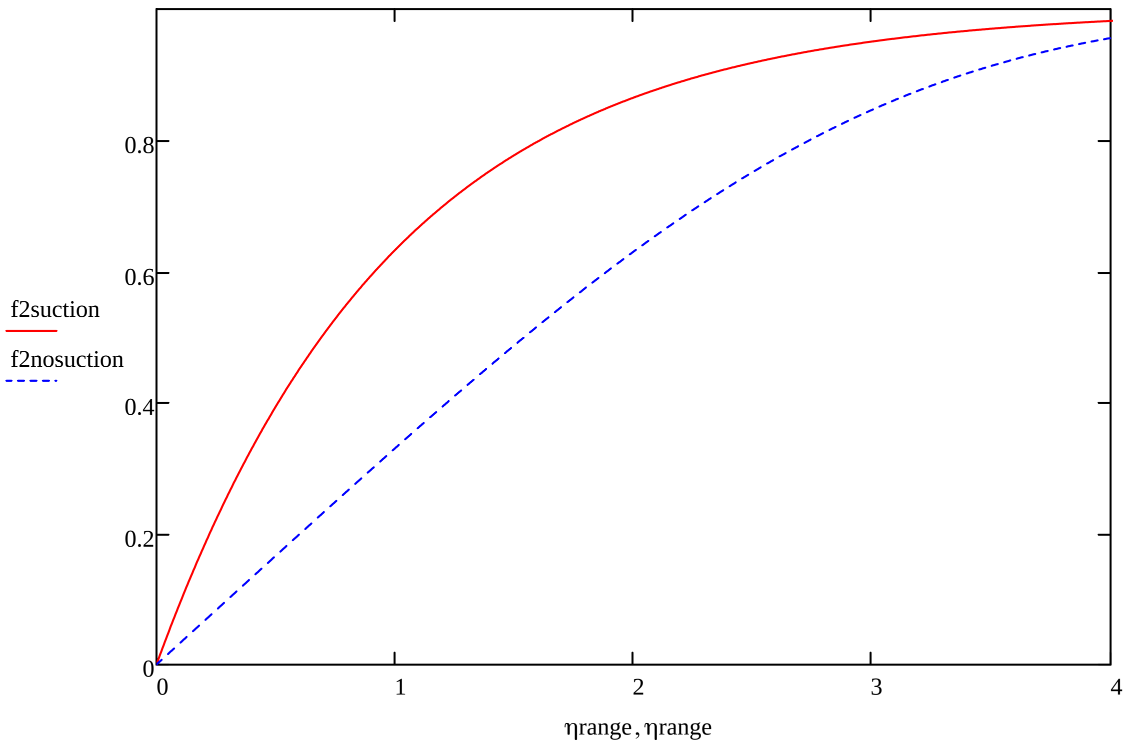
$Cd = 4.72 \times 10^{-3}$

Boundary Layer Suction

$\eta range := Zfinal^{\langle 1 \rangle}$

$f2suction := 1 - e^{-\eta range}$ $\qquad f2nosuction := (Zfinal)^{\langle 3 \rangle}$

Velocity distribution in the boundary layer with and without suction



Boundary Layer Thickness

Displacement Thickness & Momentum Thickness

$ORIGIN := 1$

Let x range from 0 to L

$begin := 0 \qquad end := L$

$NP := 100$

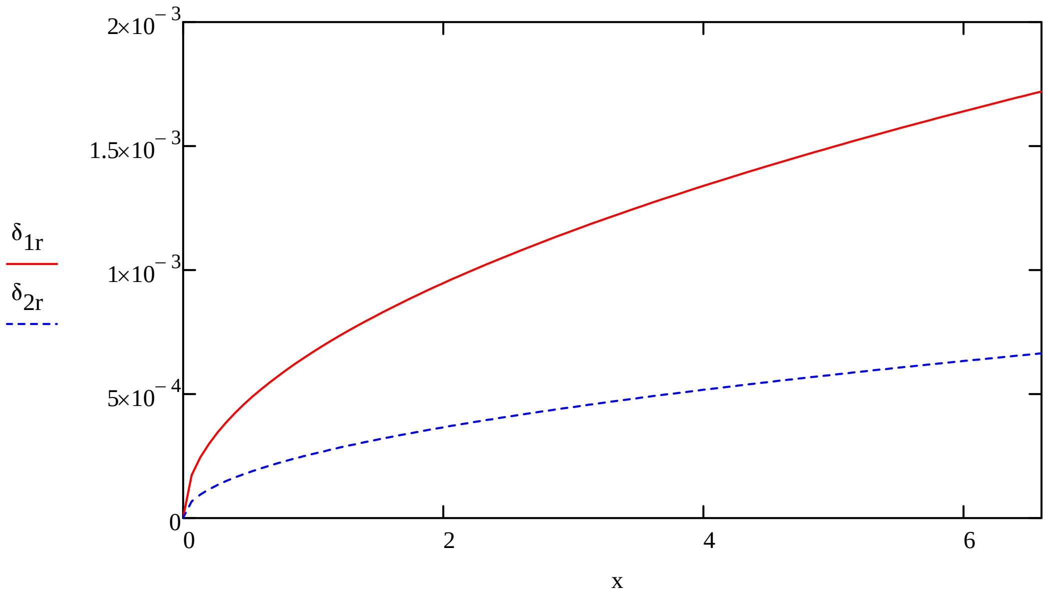
$$inc := \frac{(end - begin)}{(NP - 1)}$$

$i := 1 .. NP$

$x_i := begin + (i - 1) \cdot inc$

$$\delta_{1r} := 1.7208 \cdot \sqrt{\frac{(\upsilon \cdot x)}{U}} \qquad \delta_{2r} := 0.664 \cdot \sqrt{\frac{(\upsilon \cdot x)}{U}}$$

Displacement and Momentum Thickness variation along length of plate for u/U=0.99



Suction Design Variables

Length of applied suction from the leading edge (m) $s := xcr$

Suction Velocity (m/s), vo

Let vo/U range from 0 to 0.002

$begin := 0.026$ $end := 0.52$

$NP := 100$

$$inc := \frac{(end - begin)}{(NP - 1)}$$

$i := 1 .. NP$

$vo_i := begin + (i - 1) \cdot inc$

$$\xi := \left(\frac{-vo}{U}\right)^2 \cdot \frac{(U \cdot L)}{\upsilon} \qquad \delta_{2s} := \frac{\upsilon}{2 \cdot vo} \qquad \delta_2 := \delta_{2s_{NP}}$$

$\delta_2 = 3.783 \times 10^{-5}$

	1
1	0.436
2	0.62
3	0.835
4	[illegible]

	1
1	$7.566 \cdot 10^{-4}$
2	$6.347 \cdot 10^{-4}$
3	$5.467 \cdot 10^{-4}$
4	[illegible]

$\xi =$

4	1.083
5	1.363
6	1.675
7	2.019
8	2.395
9	2.804
10	3.244
11	3.717
12	4.222
13	4.759
14	5.328
15	5.929
16	...

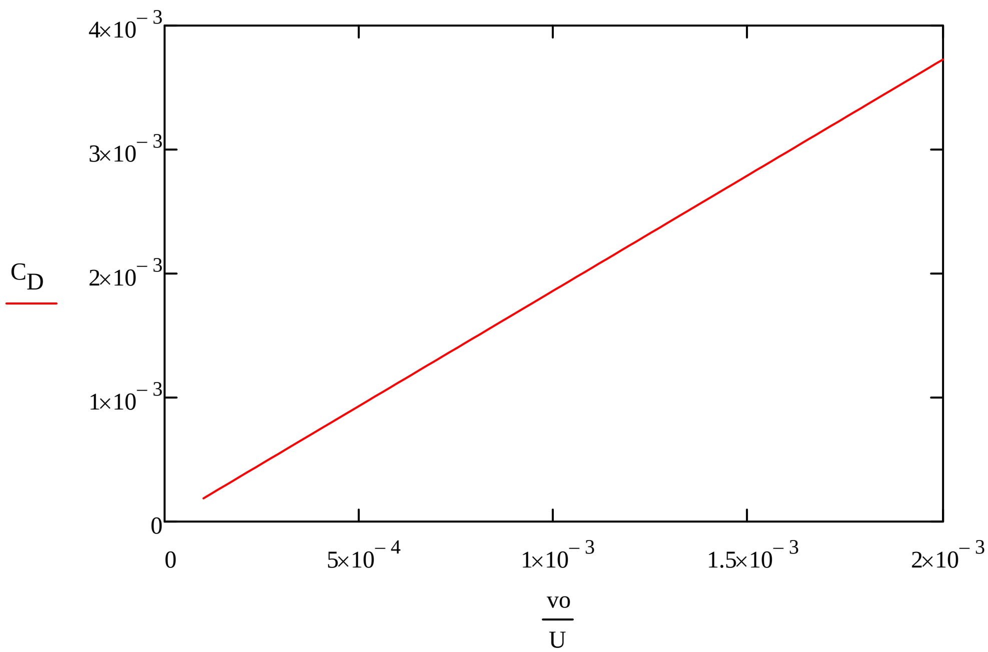
$\delta_{2s} =$

4	$4.801 \cdot 10^{-4}$
5	$4.28 \cdot 10^{-4}$
6	$3.861 \cdot 10^{-4}$
7	$3.516 \cdot 10^{-4}$
8	$3.228 \cdot 10^{-4}$
9	$2.984 \cdot 10^{-4}$
10	$2.774 \cdot 10^{-4}$
11	$2.592 \cdot 10^{-4}$
12	$2.432 \cdot 10^{-4}$
13	$2.29 \cdot 10^{-4}$
14	$2.165 \cdot 10^{-4}$
15	$2.052 \cdot 10^{-4}$
16	...

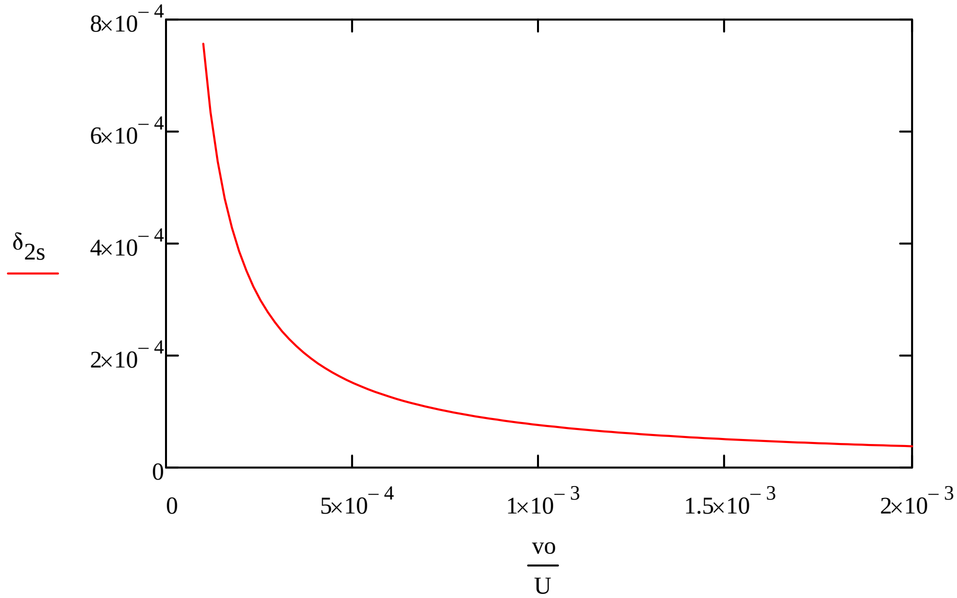
The total drag coefficient can then be calculated fom the following equation (Iglisch)

$$C_D := \frac{vo}{U}\left(\frac{1}{\xi}\,\frac{\delta_2}{\delta_{2s}} + \frac{s}{L}\right)$$

Theorectical variation in total drag coefficient against suction velocity ratio



Variation in momentum thickness with suction velocity ratio

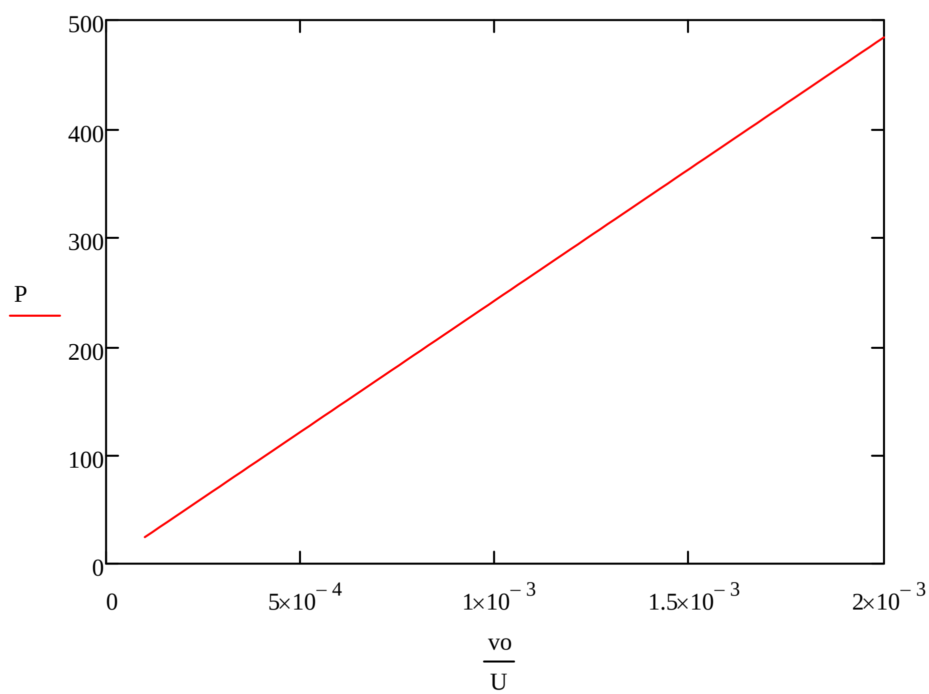


The Pump Power

The power ideally required to step up to the total head of the sucked air, per unit span, is

$$P := 0.5 \cdot \rho \cdot U^3 \cdot s \cdot \left(\frac{vo}{U}\right)$$

Variation in required pump power against suction velocity ratio



Without suction, the skin friction drag coefficient for the laminar region is:

$$Cf_l = 1.878 \times 10^{-3}$$

In the region where the flow is turbulent, no amount of suction velocity is going to be able to prevent the flow from becoming fully turbulent. It is therefore best practice to apply the suction velocity at the leading and mainatin it for at least the length of the predicted laminar region through to transistion. The aim is to prevent the transistion for laminar to turbulent. Any small increase in this value can have large impact on skin friction drag. The current laminar to turbulent transition point for the wing with no suction is:

$$xcr = 0.076$$

If suction is therefore applied accross this distance of the leading edge, the net result is that a minimum total drag is obtained at approximately the lowest suction velocity ratio at which laminar flow can survive.

For a suction velocity ratio of:

$$o := 36$$

$$\left(\frac{vo}{U}\right)_o = 7.717 \times 10^{-4}$$

$$vo_o = 0.201$$

$$\xi_o = 25.977 \qquad \delta_{2s_o} = 9.803 \times 10^{-5}$$

$$C_{D_o} = 1.438 \times 10^{-3}$$

$$P_o = 186.762$$

The Suction mass flow rate required from one pump can be calculated for the following method.

For a range of hole diameters can then be plotted

$$begin := 0.001 \qquad end := 0.01$$

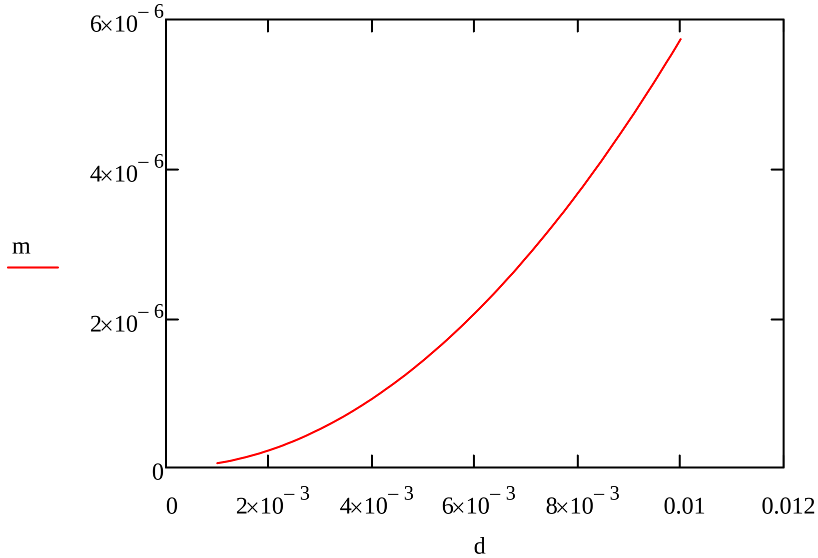
$$NP := 100$$

$$inc := \frac{(end - begin)}{(NP - 1)}$$

$$i := 1 .. NP$$

$$d_i := begin + (i - 1) \cdot inc$$

$A := \pi \frac{d^2}{4}$ $\qquad$ $m := \rho \cdot A \cdot vo_o$



For a diameter of 4mm

$o := 56$

$d_o = 6 \times 10^{-3}$

$m_o = 2.065 \times 10^{-6}$